Under what conditions when $\vec{F} = \nabla \phi$?

**Vector Identity:** $\nabla \times \nabla \phi = 0$

$\vec{F} = \nabla \phi$ iff $\nabla \times \vec{F} = 0$

Let $\vec{F} = p \hat{i} + q \hat{j} + r \hat{k}$

$\nabla \times \vec{F} = \hat{i} \left( \frac{\partial r}{\partial y} - \frac{\partial q}{\partial z} \right) + \hat{j} \left( \frac{\partial p}{\partial z} - \frac{\partial r}{\partial x} \right) + \hat{k} \left( \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right)$
\n\n\[ \nabla \times \vec{F} = 0 \quad \implies \quad \begin{align*} \frac{dr}{dy} &= \frac{dz}{dx} \\
\frac{dr}{dz} &= \frac{dx}{dy} \\
\frac{dz}{dx} &= \frac{dp}{dx} \\
\frac{dy}{dx} &= \frac{dp}{dy} \\
\end{align*} \n\]

\[ \implies F = \nabla \phi \]

\[ \implies F \cdot d\vec{r} = \nabla \phi \cdot d\vec{r} = d\phi \]

Example: \[ \vec{F} = y^2 \hat{i} + 2(xy + z) \hat{j} + 2y \hat{k} \]

Checking \( \vec{F} \):

\[ \frac{dp}{dy} = \frac{dz}{dx} \]

\[ 2y \frac{dy}{dx} = 2y \]

Yes
\[ (ii) \quad \frac{\partial p}{\partial z} = \frac{\partial r}{\partial x} = 0 \]

\[ (iii) \quad \frac{\partial g}{\partial z} = \frac{\partial r}{\partial y} = 2 \]

\[ \Rightarrow \vec{F} = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \]

\((\vec{F} = \nabla \phi)\)

To determine \(\phi\):

\[ \frac{\partial \phi}{\partial x} = p = y^2 \quad (a) \]

\[ \frac{\partial \phi}{\partial y} = q = 2(xy + z) \quad (b) \]

\[ \frac{\partial \phi}{\partial z} = r = 2y \quad (c) \]
Integrate \( \phi = y^2 x + f(y, z) \)

\[ \frac{\partial \phi}{\partial y} = 2xy + \frac{\partial f}{\partial y} \]

Compare with \( \beta \):

\[ \frac{\partial f}{\partial y} = 2z \]

\[ \implies f = 2yz + g(z) \]

\[ \phi = y^2 x + 2yz + g(z) \]

\[ \frac{\partial \phi}{\partial z} = 0 + 2y + g'(z) \]

Compare with \( \gamma \) gives:

\[ g'(z) = 0 \]

\[ \implies g(z) = c \]

\[ \phi = xy^2 + 2yz + c \]
§ Surface Integrals:

Let \( g(x, y, z) = 0 \) specify a surface \( S \).

Then
\[
\vec{n} = \pm \frac{\nabla g}{|\nabla g|}
\]

is a unit normal vector at a point on the surface.

\[ d\vec{S} = \vec{n} \, d\vec{u} \]

Surface area
Let \( \vec{F} \) be a vector defined on \( S \), then the surface integral of \( \vec{F} \) over \( S \) is given by

\[
\iint_{\partial S} \vec{F} \cdot d\vec{a}
\]

\[
= \iint_{\partial S} \vec{F} \cdot \vec{n} \, d\sigma
\]
example: Calculate $\iint_S \vec{F} \cdot d\vec{s}$

with $S: x^2 + y^2 + z^2 = 1 ; \; z \geq 0$

$\vec{F} = x \hat{i} + y \hat{j}$

$\iint_S \vec{F} \cdot \vec{n} \; d\sigma$

$\vec{n} = \frac{\nabla g}{|\nabla g|} ; \; g = x^2 + y^2 + z^2 - 1$

$= \frac{2x \hat{i} + 2y \hat{j} + 2z \hat{k}}{(2x^2 + 2y^2 + 2z^2)^{\frac{1}{2}}}$
\[ \int F \cdot n \, d\sigma = \int (x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) \cdot (x \mathbf{i} + y \mathbf{j} + z \mathbf{k}) \, d\sigma = \int (x^2 + y^2) \, d\sigma \]

**Note:**
\[ d\sigma = \frac{dx \, dy}{2} \]

\[ d\sigma (\mathbf{n} \cdot \mathbf{k}) = dx \, dy \]

\[ d\sigma \cdot \mathbf{j} = dx \, dy \]
\[
\iint (x^2 + y^2) \frac{dx \, dy}{3}
\]

Surface area
projected onto
the x-y plane

\[
3 = \sqrt{1-(x^2+y^2)}
\]

\[
\iint (x^2 + y^2) \frac{dx \, dy}{\sqrt{1-(x^2+y^2)}}
\]

Using polar coordinates:
\[
dx \, dy \rightarrow r \, dr \, d\theta
\]
\begin{align*}
\frac{2\pi}{a^2} & \int_{0}^{a} \frac{1}{\sqrt{1-r^2}} \, dr \, dr \, da \\
\text{let } u &= 1-r^2 \\
\frac{du}{dr} &= -2r \\
\frac{du}{2} &= -r \, dr \\
\sqrt{1-r^2} &= u^{1/2} \\
r^2 &= 1-u
\end{align*}

\begin{align*}
\int_{0}^{2\pi} \int_{0}^{a} \frac{(1-u)\left(-\frac{du}{2}\right)}{u^{1/2}} \, dr \, da \\
= \int_{0}^{2\pi} \int_{0}^{a} \frac{(u-1)}{2u^{1/2}} \, dr \, da \\
= \frac{2\pi}{a^2} \int_{0}^{a} \left(\frac{u-1}{2u^{1/2}}\right) \, du = \frac{2\pi}{a^2} \int_{0}^{1} \left[u^{1/2} - u^{-1/2}\right] \, du
\end{align*}
\[ \mathbf{E} \cdot \mathbf{n} = \frac{1}{3} \mathbf{n} \cdot \mathbf{u} - 2 \mathbf{u} \cdot \mathbf{n} \]

Gauss' Theorem (Divergence Theorem)

\[ \iiint \nabla \cdot \mathbf{F} \, dV = \iint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{n} \, dS \]

where \( \mathbf{F} \) is any vector field, \( \mathcal{S} \) is any surface, and \( V \) is the volume enclosed by \( \mathcal{S} \).
\[ \oint_{C} \mathbf{F} \cdot d\mathbf{r} = \iint_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, d\sigma \]

\( C \) : boundary of \( S \)