Continuity: A vector \( \mathbf{u}(t) \) is said to be continuous at \( t = t_0 \) if it is defined in some neighborhood of \( t_0 \) and \( \lim_{t \to t_0} \mathbf{u}'(t) = \mathbf{u}'(t_0) \)

\[
\mathbf{u}(t) = u_1(t) \mathbf{i} + u_2(t) \mathbf{j} + u_3(t) \mathbf{k}
\]

then \( \mathbf{u}(t) \) is continuous iff its three components are continuous.
**Differentiability:** A vector field \( \vec{U}(t) \) is said to be differentiable at a point \( t_0 \) if the limit

\[
\lim_{\Delta t \to 0} \frac{\vec{U}(t_0 + \Delta t) - \vec{U}(t_0)}{\Delta t}
\]

exists.

\[= \vec{U}'(t_0) \]

is called the derivative of \( \vec{U}'(t) \) at \( t_0 \).

\[(i) \quad (C \vec{U})' = C \vec{U}' \]

\[(ii) \quad (\vec{U} + \vec{V})' = \vec{U}' + \vec{V}' \]

\[(iii) \quad (\vec{U} \times \vec{V})' = \vec{U}' \times \vec{V} + \vec{U} \times \vec{V}' \]

\[(iv) \quad (\vec{U} \cdot \vec{V})' = \vec{U}' \cdot \vec{V} + \vec{U} \cdot \vec{V}' \]
Gradient Vector

A scalar field: \( f(x, y, z) \)

\[ \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \] are the rates of change of \( f \) in \( x, y \) and \( z \) directions.

\( \vec{b} \) (unit vector)

\[ \frac{df}{ds} = ? \] along \( \vec{b} \)

The directional derivative \( \nabla f(x, y, z) \) in the direction of \( \vec{b} \)
\[
\frac{df}{ds} = \lim_{s \to 0} \frac{f(A) - f(P)}{s}
\]

\[
\vec{a} + s \vec{b} = \vec{r} = x(s) \vec{i} + y(s) \vec{j} + z(s) \vec{k}
\]

\[
f = f(x, y, z) = f(x(s), y(s), z(s))
\]

\[
\frac{df}{ds} = \frac{df}{dx} \frac{dx}{ds} + \frac{df}{dy} \frac{dy}{ds} + \frac{df}{dz} \frac{dz}{ds}
\]

\[
\frac{d\vec{r}}{ds} = \frac{dx}{ds} \vec{i} + \frac{dy}{ds} \vec{j} + \frac{dz}{ds} \vec{k} = \vec{b}
\]

\[
(= \frac{d}{ds}(\vec{a} + s \vec{b}) = \vec{b})
\]
Introduce: $\nabla f = \text{grad } f$

$$= \frac{df}{dx} \hat{i} + \frac{df}{dy} \hat{j} + \frac{df}{dz} \hat{k}$$

Comparing gives:

$$\frac{df}{ds} = \nabla f \cdot \frac{dr}{ds} = \nabla f \cdot \vec{b}$$

e.g. in a 1-D

$$f = f(x)$$

$$\nabla f = \frac{df}{dx} \hat{i} \quad \frac{df}{ds} = \frac{df}{dx} \hat{i}$$

$$\vec{b} = \hat{i}$$
Example: Find the directional derivative \( \frac{\partial f}{\partial s} \) of \( f(x, y, z) = 2x^2 + 3y^2 + z^2 \) at the point \((2, 1, 3)\) in the direction \( \mathbf{a} = \mathbf{i} - 2\mathbf{k} \)

\[ \nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \]
\[ = 4x \mathbf{i} + 6y \mathbf{j} + 2z \mathbf{k} \]

at \((2, 1, 3)\) we have
\[ \nabla f = 8 \mathbf{i} + 6 \mathbf{j} + 6 \mathbf{k} \]

\[ \frac{\partial f}{\partial s} = \nabla f \cdot \mathbf{b} \]
\[ \vec{b} = \frac{\vec{a}}{|\vec{a}|} = \frac{i - 2k}{\sqrt{5}} \]

\[ \therefore \frac{df}{ds} = (8 \vec{i} + 6 \vec{j} + 6 \vec{k}) \cdot (\vec{i} - 2\vec{k})/\sqrt{5} \]

\[ = \frac{8 - 12}{\sqrt{5}} = -\frac{4}{\sqrt{5}} \]

**Some Properties of the Gradient Vector**

Consider a surface \( S \) represented by \( f(x, y, z) = c \)

with \( c \) assuming different values, we have a family of surfaces, which are called the level surface \( S_f \).
\[ f = xyz = c \]
\[ xyg = c_1 \]
\[ xyg = c_2 \]
\[ xyg = c_3 \]
\[ f = x^2 + y^2 + z^2 = c_1 \]
Let \( l \) be a curve in space which lies on \( S \), represented by \( \vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} + z(t) \hat{k} \).

We have

\[ f(x, y, z) = c \]

\[ f(x(t), y(t), z(t)) = c \]

\[ \Rightarrow \quad \frac{df}{dt} = 0 \]

\[ \Rightarrow \quad \frac{df}{dx} \frac{dx}{dt} + \frac{df}{dy} \frac{dy}{dt} + \frac{df}{dz} \frac{dz}{dt} = 0 \]
\[ \nabla f \cdot \mathbf{r} = 0 \]

\[ \mathbf{r} = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \]

But \( \mathbf{r} \) is tangent to \( C \)

\[ + \nabla f \perp \mathbf{r} \]

\( \nabla f \) is normal to the level surface.

\[ \frac{d}{dt} \mathbf{r} = -k \nabla f \]

\( \nabla f \) has the direction of maximum increase of \( f \).