Ordinary Differential Equations

§ Basic Nomenclature

\[ a_0(y, t) \frac{d^n y}{dt^n} + a_1(y, t) \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_{n-1}(y, t) \frac{dy}{dt} + a_n(y, t) y = f(t) \]

\[ t : \text{indep variable} \]
\[ y : \text{dependent variable} \]

- If \( f(t) = 0 \); the equation is homogeneous
- If \( f(t) \neq 0 \); " " nonhomogeneous
n is the order of the D.E.; the order of the highest derivative.

The equation is linear if a’s all depend only on it, or are constants.

The equation is ordinary if it contains only ordinary derivatives.

\[ \frac{d^2y}{dx^2} + \frac{dy}{dx} + 3y = 2t \quad (O. \text{D.E.}) \]

\[ \frac{dy}{dt} = \frac{d^2y}{dx^2} \quad (P. \text{D.E.}) \]

A solution is any functional relation that satisfies the D.E. in a form not containing integrals or derivatives.
of unknown sets.

Solution of 1st order equations

For $n = 1$, the homogeneous equation is:

$$a_0(y+t) \frac{dy}{dt} + a_1(y+t)y = 0$$

Rearrange and renameing things:

$$F(x, y) dx + G(x, y) dy = 0$$

1. Exact Differential

Consider $U(x, y) = C$

$$\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy = 0$$
Comparing the two equations:
\[ F(x, y) = \frac{\partial U}{\partial x} \quad \text{and} \quad G(x, y) = \frac{\partial U}{\partial y} \]

Note:
\[ \frac{\partial F}{\partial y} = \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial G}{\partial x} = \frac{\partial^2 U}{\partial x \partial y} \]

If it is an exact differential, then \( \frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} \) and we can integrate \( \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} \) to get \( U \).
e.g. Find a solution to
\[
\frac{2xy + 1}{y} \, dx + \frac{y - x}{y^2} \, dy = 0
\]

\[\frac{\partial F}{\partial y} = -\frac{1}{y^2}\]

\[\frac{\partial G}{\partial x} = -\frac{1}{y^2}\]

\[\text{Yes}\]

\[\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}\]

\[\therefore \text{it is an exact differential}\]

\[F = \frac{\partial U}{\partial x}\]

\[G = \frac{\partial U}{\partial y}\]

\[\Rightarrow U = \int F \, dx + f_1(y)\]

\[U = \int G \, dy + f_2(x)\]
\[ U = \int \frac{2xy+1}{y} \, dx + f_1(y) \quad U = \int \frac{y-x}{y^2} \, dy + f_2(x) \]
\[ = x^2y + x + f_1(y) \quad = \ln y + \frac{x}{y} + f_2(x) \]
\[ = x^2 + \frac{x}{y} + f_1(y) \]

Comparing both sides:
\[ f_1(y) = \ln y \quad f_2(x) = x^2 \]

\[ U = x^2 + \frac{x}{y} + \ln y \quad U = \ln y + \frac{x}{y} + x^2 \]

\[ \Rightarrow x^2 + \frac{x}{y} + \ln y = C \]
Use of an integrating factor

If \( F(x,y)dx + G(x,y)dy \) is not an exact differential (i.e. \( \frac{\partial F}{\partial y} \neq \frac{\partial G}{\partial x} \))

it can be made one by multiplying by an integrating factor, \( p(x) \).

\[ p(x)F(x,y)dx + p(x)G(x,y)dy \]

may then be an exact differential.

E.g. Consider \( x \, dy - y \, dx \)

\[ \frac{\partial F}{\partial x} \]
\[ F = -y \quad \text{and} \quad G = x \]
\[ \frac{dF}{dy} = -1 \neq \frac{dG}{dx} = 1 \]
\[ \therefore \text{not an exact differential} \]

**Observe:**
\[
\begin{align*}
\frac{d}{dx} \left( \frac{y}{x} \right) &= \frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} \\
&= \frac{1}{x} \frac{dF}{dy} - \frac{F}{x^2} \\
&= \frac{1}{x} \frac{dF}{dy} - \frac{1}{x^2} \\
&= \frac{x}{x^2} \frac{dF}{dy} - \frac{y}{x^2} \\
&= \frac{y}{x} \frac{dF}{dy} - \frac{y}{x^2} dx \\
&= x dy - y dx
\end{align*}
\]

\[ \therefore \text{If we multiply } x^2 \text{ by the integrating factor } \frac{1}{x^2}, \text{ the} \]
result is an exact differential

we have \( \frac{d}{dx} \left( \frac{y}{x} \right) = 0 \)

\[ \Rightarrow \quad \frac{y}{x} = c \quad \Rightarrow \quad y = cx \]

\section*{Separation of Variable}

\[ F(x, y) \, dx + G(x, y) \, dy = 0 \]

\[ x \, dx - y \, dy \]

If \( F \) and \( G \) are separable

i.e.

\[ F(x, y) = F_1(x) \, F_2(y) \]
\[ G(x, y) = G_1(x) \, G_2(y) \]

then the solution is simple.
\[ F_1(x)F_2(y) \, dx + G_1(x)G_2(y) \, dy = 0 \]

\[ \Rightarrow \quad \frac{F_1(x)}{G_1(x)} \, dx + \frac{G_2(y)}{F_2(y)} \, dy = 0 \]

\[ \Rightarrow \quad \int \frac{F_1(x)}{G_1(x)} \, dx + \int \frac{G_2(y)}{F_2(y)} \, dy = C \]

\underline{Change of Variables}

\textbf{Defn.} A function \( F(x,y) \) is homogeneous to degree \( n \) if there is a constant \( n \), s.t. for every \( \lambda \)

\[ F(\lambda x, \lambda y) = \lambda^n F(x,y) \]

\text{e.g.} \( F(x,y) = x^6 + 2x^2y^4 + xy^5 \)
If \( F(x,y) \) and \( G(x,y) \) are both homogeneous to degree \( n \), then the substitution \( y = vx \) or \( x = uy \) leads to a separable equation.

Let \( y = vx \) \( \Rightarrow \) \( dy = v \, dx + x \, dv \)

D.E. becomes:

\[
F(x, vx) \, dx + G(x, vx) \, dv - [v \, dx + x \, dv] = 0
\]

\( x^n \) \( F(1, v) \, dx + x^n \, G(1, v) [v \, dx + x \, dv] = 0 \)

\( \Rightarrow [F(1, v) + v \, G(1, v)] \, dx = -x \cdot G(1, v) \, dv \)
\[ \Rightarrow \int \frac{dy}{y} = -\int \frac{dG(y)}{F(y) + vG(y)} + C \]

\[ \Rightarrow \ln x = \ln v + C \]

e.g. Solve \((y^2 - xy)dx + x^2 dy = 0\)

\[ F = y, \quad G = y^2 - x \]

\[ \frac{dF}{dy} = 2y - x \neq \frac{dG}{dx} = 2x \]

\[ \text{:: not an exact differential} \]

By inspection, \(F + G\) are homogeneous.

Hence \(n = 2\); let \(y = vx\)

\[(v^2x^2 - xvx)dx + x^2[2vdx + xdv] = 0\]

\[x^2(v^2 - v)dx + x^2[2vdx + xdv] = 0\]
\[ x^2 \, v^2 \, dx = -x^3 \, dv \]

\[ \Rightarrow \int \frac{dx}{x} = -\int \frac{dv}{v^2} + C \]

\[ \Rightarrow \ln x = \frac{1}{v^2} + C \]

\[ \Rightarrow \ln x = \frac{x}{y} + C \]

\[ (y = vx, \quad v = \frac{y}{x}) \]

\[ \text{§ Linear Equations of First Order} \]

\[ a_0(x) \frac{dy}{dx} + a_1(x) \, y = g(x) \]

\[ \Rightarrow a_0(x) : \]

\[ \left( \frac{dy}{dx} + a_1(x) \, y = f(x) \right) \]
Multiply by an integrating factor $p(x)$

$$p(x) \frac{dy}{dx} + p(x) q(x) y = p(x) f(x) \quad (1)$$

Compare with the exact differential

$$\frac{d}{dx}[p(x) y] = p(x) \frac{dy}{dx} + y \frac{dp}{dx}$$

L.H.S. of (1) will be an exact differential if:

$$\frac{dp}{dx} = p(x) q(x)$$

$$\implies p(x) = e^{\int q(x) dx}$$

Eq (1) becomes:


\[ p(x) \frac{dy}{dx} + p(x) q(x)y = \frac{d}{dx}[p(x)y] = p(x)f(x) \]

\[ \Rightarrow p(x)y = \int p(x)f(x) \, dx + C \]

\[ \Rightarrow y = \frac{1}{p(x)} \left[ \int p(x)f(x) \, dx + C \right] \]

- Nonhomogeneous solution
- Homogeneous solution
- (particular solution)

\[ y = e^{-\int q(x) \, dx} \int e^{\int q(x) \, dx} f(x) \, dx + C e^{-\int q(x) \, dx} \]
\[ e^y \cdot x dy - y dx = 0 \]

\[ \Rightarrow y = cx \]

\[ \frac{dy}{dx} - \frac{y}{x} = 0 \]

\[ q_1 = -\frac{1}{x}, \quad f(x) = 0 \]

\[ p(x) = e^{\int q_1 dx} = e^{-\ln x} = \frac{1}{x} \]

\[ \frac{dy}{dx} - \frac{y}{x} = 0 \]

\[ \Rightarrow \frac{d}{dx} \left( \frac{y}{x} \right) = 0 \]

\[ \Rightarrow \frac{y}{x} = c \]