$$= \frac{1}{r^2 \sin\phi}\left[\frac{\partial}{\partial r}\left(r^2 \sin\phi\, F_r\right) + \frac{\partial}{\partial \phi}\left(r \sin\phi\, F_\phi\right) + \frac{\partial}{\partial \theta}\left(r F_\theta\right)\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\, F_\phi\right) + \frac{1}{r\sin\phi}\frac{\partial F_\theta}{\partial \theta}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$$

$$\nabla \times \vec{F} = \frac{1}{r^2 \sin\phi} \begin{vmatrix} \vec{u}_r & r\vec{u}_\phi & r\sin\phi\,\vec{u}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ F_r & rF_\phi & r\sin\phi\,F_\theta \end{vmatrix}$$

§ Worked Example on the Divergence Th$^{\underline{m}}$

(taken from Hildebrand : #73 on P. 331)

~~Det~~ Determine $\iint_S \vec{F}\cdot\vec{n}\,d\sigma$ by the Divergence Th$^{\underline{m}}$

(a) $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$

$S = x^2 + y^2 + z^2 = 1$

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_V \nabla\cdot\vec{F}\,d\tau$$

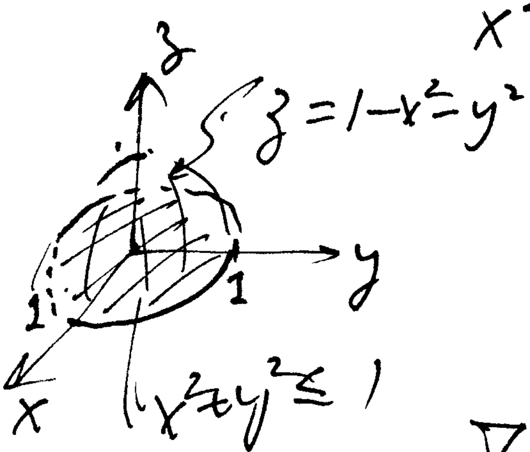
$$\nabla\cdot\vec{F} = 1+1+1 = 3$$

$$\therefore \iiint_V \nabla\cdot\vec{F}\,d\tau = 3\iiint_V d\tau$$

$$= 3\left(\frac{4}{3}\pi r^3\right)\Big|_{r=1} = 4\pi$$

(b) $\vec{F} = xy\,\vec{i} + xz\,\vec{j} + (1-z-yz)\,\vec{k}$

$S: \quad z = 1-x^2-y^2 \ ; \ z \geq 0$

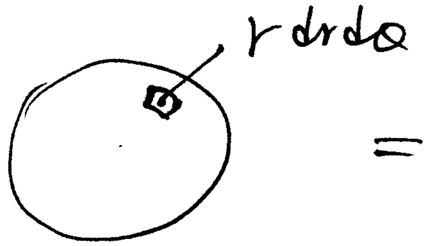


$x^2+y^2 \leq 1 \quad ; z=0$

$$\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_V \nabla\cdot\vec{F}\,d\tau$$

$$\nabla\cdot\vec{F} = y + 0 + (-1-y)$$
$$= -1$$

$$-\iiint_V d\tau = -\iint_{\substack{\text{bottom}\\ \text{disc } (r\leq 1)}} z\, r\,dr\,d\theta$$



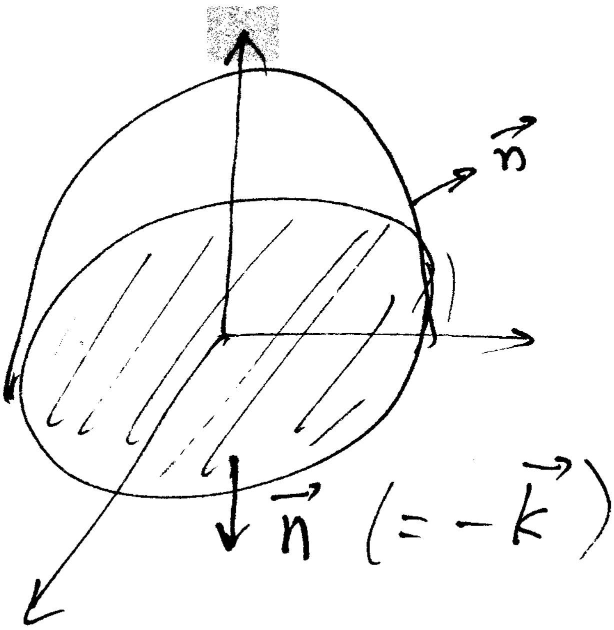
$$= -\int_0^{2\pi}\int_0^1 (1-r^2)\, r\,dr\,d\theta$$

$$= -2\pi \int_0^1 (1-r^2)\, r\, dr$$

$$= -2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = -\frac{\pi}{2}$$

(c) $$\underset{\substack{\text{top}\\\text{curve}\\\text{surface}}}{\iint} \vec{F}\cdot\vec{n}\, dS = \underset{\substack{\text{closed}\\\text{surface}}}{\iint} - \underset{\substack{\text{bottom}\\\text{disc}}}{\iint}$$

$$\underset{\text{closed surface}}{\iint} = -\frac{\pi}{2}$$



$$\iint_{\text{bottom disc}} \vec{F}\cdot\vec{n}\,d\sigma$$

$$= \iint \left[xy\,\vec{i} + xz\,\vec{j} + (1 - z - yz)\,\vec{k} \right]\cdot(-\vec{k})\,d\sigma$$

$$= \iint -(1 - z - yz)\,d\sigma$$

along the bottom disc $z = 0$

$$= \iint - \, d\sigma = - \iint_{\substack{\text{bottom}\\ \text{disc}}} d\sigma = -\pi$$

$$\therefore \iint_{\text{top}} \vec{F} \cdot \vec{n} \, d\sigma = \left(-\frac{\pi}{2}\right) - (-\pi)$$

$$= \frac{\pi}{2}.$$