$$df = \nabla f \cdot d\vec{r}$$

$$df = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2 + \frac{\partial f}{\partial u_3} du_3$$

& $$d\vec{r} = \vec{U_1}\, du_1 + \vec{U_2}\, du_2 + \vec{U_3}\, du_3$$

$$= h_1 \vec{u_1}\, du_1 + h_2 \vec{u_2}\, du_2 + h_3 \vec{u_3}\, du_3$$

Let $\nabla f = \lambda_1 \vec{u_1} + \lambda_2 \vec{u_2} + \lambda_3 \vec{u_3}$

then

$$\nabla f \cdot d\vec{r} = \lambda_1 h_1\, du_1 + \lambda_2 h_2\, du_2 + \lambda_3 h_3\, du_3$$

Comparing with df gives:

$$\lambda_1 h_1 = \frac{\partial f}{\partial u_1} \Longrightarrow \lambda_1 = \frac{1}{h_1}\frac{\partial f}{\partial u_1}$$

$$\lambda_2 h_2 = \frac{\partial f}{\partial u_2} \Longrightarrow \lambda_2 = \frac{1}{h_2}\frac{\partial f}{\partial u_2}$$

$$\& \quad \lambda_3 h_3 = \frac{\partial f}{\partial u_3} \Longrightarrow \lambda_3 = \frac{1}{h_3}\frac{\partial f}{\partial u_3}$$

$$\therefore \quad \nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\vec{u}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\vec{u}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\vec{u}_3$$

$$= \frac{\partial f}{\partial s_1}\vec{u}_1 + \frac{\partial f}{\partial s_2}\vec{u}_2 + \frac{\partial f}{\partial s_3}\vec{u}_3$$

(because : $ds_i = h_i\, du_i$)

$$\nabla = \vec{u}_1\frac{\partial}{\partial s_1} + \vec{u}_2\frac{\partial}{\partial s_2} + \vec{u}_3\frac{\partial}{\partial s_3}$$

$$\text{or} = \vec{u}_1\frac{1}{h_1}\frac{\partial}{\partial u_1} + \vec{u}_2\frac{1}{h_2}\frac{\partial}{\partial u_2} + \vec{u}_3\frac{1}{h_3}\frac{\partial}{\partial u_3}$$

e.g. cylindrical coordinates:

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\} \quad \begin{aligned} u_1 &= r \\ u_2 &= \theta \\ u_3 &= z \end{aligned}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$
$$= r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + z\vec{k}$$

$$\vec{U}_1 = \frac{\partial \vec{r}}{\partial u_1} = \frac{\partial \vec{r}}{\partial r} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

$$\vec{U}_2 = \frac{\partial \vec{r}}{\partial u_2} = \frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta\,\vec{i} + r\cos\theta\,\vec{j}$$

$$\vec{U}_3 = \frac{\partial \vec{r}}{\partial u_3} = \frac{\partial \vec{r}}{\partial z} = \vec{k}$$

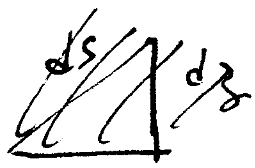
$$h_1 = h_r = |\vec{U}_1| = |\cos\theta\, \vec{i} + \sin\theta\, \vec{j}|$$
$$= 1$$
$$h_2 = h_\theta = |\vec{U}_2| = |-r\sin\theta\, \vec{i} + r\cos\theta\, \vec{j}|$$
$$= r$$
$$h_3 = h_z = |\vec{U}_3| = |\vec{k}|$$
$$= 1$$

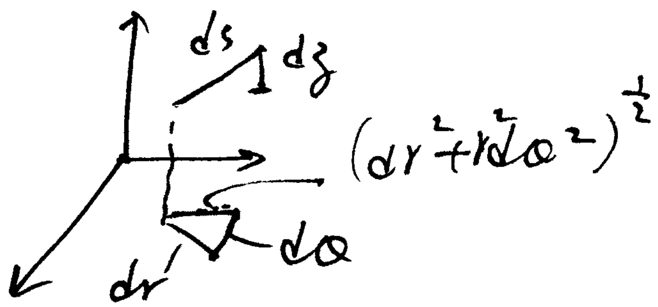
It can be shown easily that

$$\left.\begin{aligned} \vec{u}_r &= \cos\theta\, \vec{i} + \sin\theta\, \vec{j} \\ \vec{u}_\theta &= -\sin\theta\, \vec{i} + \cos\theta\, \vec{j} \end{aligned}\right\} \quad \vec{u}_1 \cdot \vec{u}_\theta = 0$$
$$\vec{u}_z = \vec{k}$$

line element :

$$ds^2 = h_r^2\, dr^2 + h_\theta^2\, d\theta^2 + h_z^2\, dz^2$$
$$= dr^2 + r^2 d\theta^2 + dz^2$$





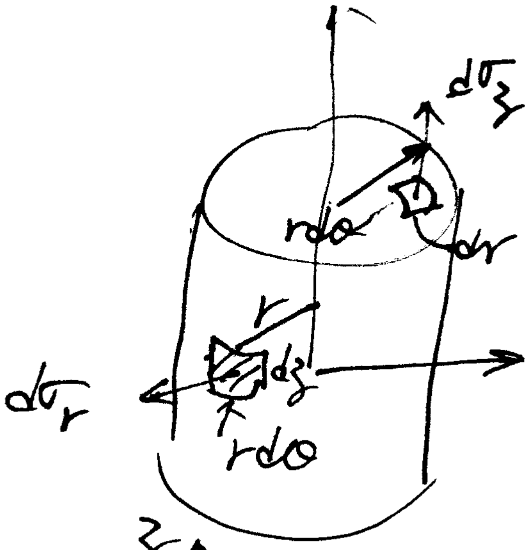
Area Element

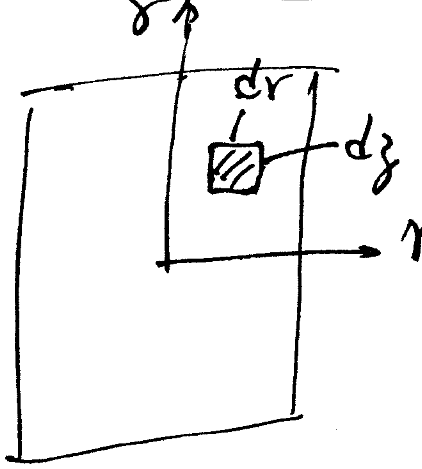
$$d\sigma_1 = h_2 h_3 \, du_2 \, du_3$$

$$\Rightarrow d\sigma_r = h_\theta h_z \cancel{du_\theta} \, d\theta \, dz$$
$$= r \, d\theta \, dz$$

$$d\sigma_\theta = h_r h_z \, dr \, dz$$
$$= dr \, dz$$

$$d\sigma_z = h_r h_\theta \, dr \, d\theta$$
$$= r \, dr \, d\theta$$





Volume Element:

$$d\tau = h_1 h_2 h_3 \, du_1 \, du_2 \, du_3$$
$$= r \, dr \, d\theta \, dz$$

Gradient:

$$\nabla = \vec{u}_r \frac{\partial}{\partial r} + \vec{u}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \vec{u}_z \frac{\partial}{\partial z}$$

Divergence

$$\nabla \cdot \vec{F} \left(= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \right)$$

Let $\vec{F} = F_1 \vec{u}_1 + F_2 \vec{u}_2 + F_3 \vec{u}_3$

$$\nabla \cdot \vec{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1}(h_2 h_3 F_1) + \frac{\partial}{\partial u_2}(h_1 h_3 F_2) \right.$$

$$+ \frac{\partial}{\partial u_3}(h_1 h_2 F_3)\Big]$$

In cylindrical coordinates (r, θ, z)

$$\vec{F} = F_r \vec{u}_r + F_\theta \vec{u}_\theta + F_z \vec{u}_z$$

$$\nabla \cdot \vec{F} = \frac{1}{h_r h_\theta h_z}\Big[\frac{\partial}{\partial r}(h_\theta h_z F_r) + \frac{\partial}{\partial \theta}(h_r h_z F_\theta)$$

$$+ \frac{\partial}{\partial z}(h_r h_\theta F_z)\Big]$$

$$= \frac{1}{r}\Big[\frac{\partial}{\partial r}(r F_r) + \frac{\partial F_\theta}{\partial \theta} + \frac{\partial (r F_z)}{\partial z}\Big]$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r F_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

e.g. For incompressible flow :

$$\nabla \cdot \vec{V} = 0$$

(x, y, z) : $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

$(\vec{V} = u\vec{i} + v\vec{j} + w\vec{k})$

(r, θ, z) : $\vec{V} = (V_r, V_\theta, V_z)$

$$\nabla \cdot \vec{V} = \frac{1}{r}\frac{\partial}{\partial r}(r V_r) + \frac{\partial V_\theta}{r\partial\theta} + \frac{\partial V_z}{\partial z} = 0$$

∇^2 (Laplacian Operator) $\equiv \nabla \cdot \nabla$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (x, y, z)$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial u_3}\right)\right]$$

In (r, θ, z) :

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Note : the expressions given above for ∇^2 are valid for scalar functions only :

$$\nabla^2 \vec{V} \neq \nabla^2 u_r \vec{r} + \nabla^2 u_\theta \vec{\theta} + \nabla^2 u_z \vec{z}$$

$$\nabla^2 \vec{V} = \nabla(\nabla \cdot \vec{V}) - \nabla \times (\nabla \times \vec{V})$$

$$\left(\nabla^2 \vec{V} = \nabla^2 u \, \vec{i} + \nabla^2 v \, \vec{j} + \nabla^2 w \, \vec{k} \right)$$

e.g. Navier-Stokes Eqn:

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla p + \nu(\nabla^2 \vec{V})$$

If (x, y, z) coordinate system is used then

$$\frac{\partial u}{\partial t} + (\vec{V} \cdot \nabla) u = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \nabla^2 u$$

$$\frac{\partial v}{\partial t} + (\vec{V} \cdot \nabla) v = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \nabla^2 v$$

$$\frac{\partial w}{\partial t} + (\vec{V} \cdot \nabla) w = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 w$$

$$\vec{u}_r\left(\frac{1}{r}\frac{\partial F_z}{\partial \theta}-\frac{\partial F_\theta}{\partial z}\right)$$

$$\vec{u}_\theta\left(\frac{\partial F_r}{\partial z}-\frac{\partial F_z}{\partial r}\right)$$

$$\vec{u}_z\left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta)-\frac{1}{r}\frac{\partial F_r}{\partial \theta}\right)$$