Under what conditions when $\vec{F} = \nabla\phi$?

$$\boxed{\text{Vector identity} : \quad \nabla \times \nabla\phi \equiv 0}$$

$$\vec{F} = \nabla\phi \quad \text{iff} \quad \nabla \times \vec{F} = 0$$

let $\vec{F} = p\,\vec{i} + q\,\vec{j} + r\,\vec{k}$

$$\nabla \times \vec{F} = \vec{i}\left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right) + \vec{j}\left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right) + \vec{k}\left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right)$$

$$\nabla \times \vec{F} = 0 \implies \boxed{\begin{aligned} \frac{\partial r}{\partial y} &= \frac{\partial q}{\partial z} \\ \frac{\partial p}{\partial z} &= \frac{\partial r}{\partial x} \\ \& \quad \frac{\partial q}{\partial x} &= \frac{\partial p}{\partial y} \end{aligned}} \implies \vec{F} = \nabla \phi$$

$$\implies \vec{F} \cdot d\vec{r} = \nabla\phi \cdot d\vec{r} = d\phi$$

<u>Example</u> : $\vec{F} = \underbrace{y^2}_{p}\,\vec{i} + \underbrace{2(xy+z)}_{q}\,\vec{j} + \underbrace{2y}_{r}\,\vec{k}$

<u>Checking</u> ~~for~~ (i) : $\underbrace{\frac{\partial p}{\partial y}}_{2y} \overset{?}{=} \frac{\partial q}{\partial x} = 2y$ Yes

(ii) $\underset{0}{\frac{\partial p}{\partial z}} \overset{?}{=} \underset{0}{\frac{\partial r}{\partial x}}$ Yes

(iii) $\underset{2}{\frac{\partial q}{\partial z}} \overset{?}{=} \underset{2}{\frac{\partial r}{\partial y}}$ Yes

$$\Longrightarrow \vec{F} = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

$(\vec{F} = \nabla \phi)$

To determine ϕ :

$$\frac{\partial \phi}{\partial x} = p = y^2 \quad (a)$$

$$\frac{\partial \phi}{\partial y} = q = 2(xy+z) \quad (b)$$

$$\frac{\partial \phi}{\partial z} = r = 2y \quad (c)$$

Integrate (a) : $\phi = y^2 x + f(y, z)$

$\Longrightarrow \frac{\partial \phi}{\partial y} = 2xy + \frac{\partial f}{\partial y}$

Compare with (b) ; $\frac{\partial f}{\partial y} = 2z$

$\Longrightarrow f = 2yz + g(z)$

$\therefore \phi = y^2 x + 2yz + g(z)$

$\frac{\partial \phi}{\partial z} = 0 + 2y + g'(z)$

Compare with (c) gives: $g'(z) = 0$

$\Longrightarrow g(z) = C$

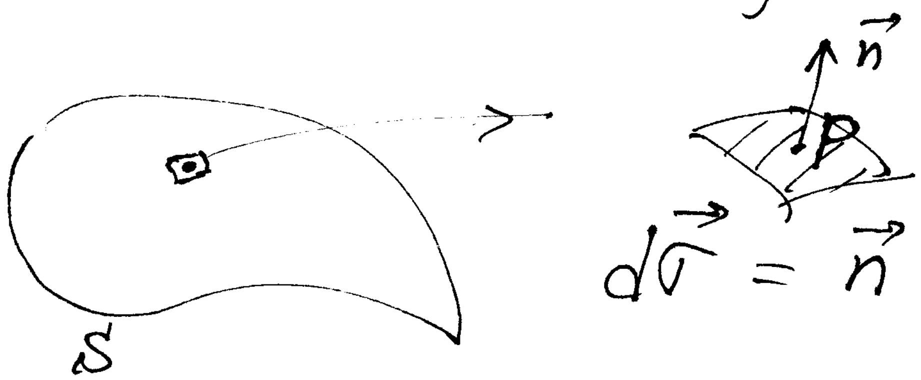
$$\boxed{\therefore \phi = xy^2 + 2yz + C}$$

§ <u>Surface Integrals</u> :

Let $g(x, y, z) = 0$ specify a surface S

then
$$\vec{n} = \pm \frac{\nabla g}{|\nabla g|}$$

is a unit normal vector at a point on the surface.

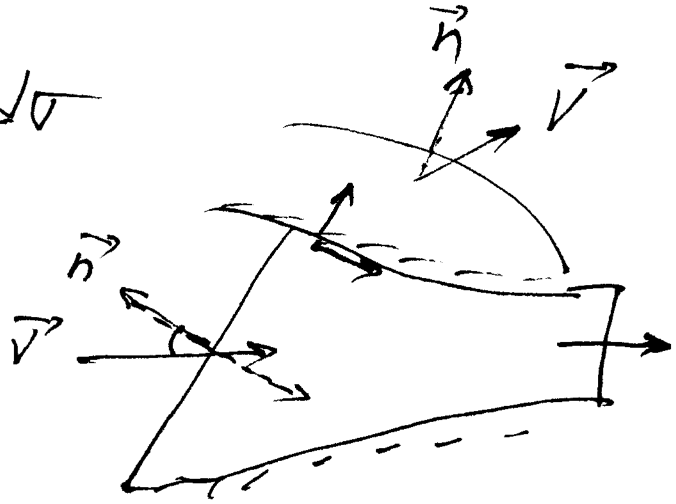


$$d\vec{\sigma} = \vec{n}\, d\sigma$$ surface area

Let $\vec{F}$ be a vector defined on S, then the surface integral of $\vec{F}$ over S is given by

$$\iint_S \vec{F} \cdot d\vec{\sigma}$$

$$= \iint_S \vec{F} \cdot \vec{n}\, d\sigma$$

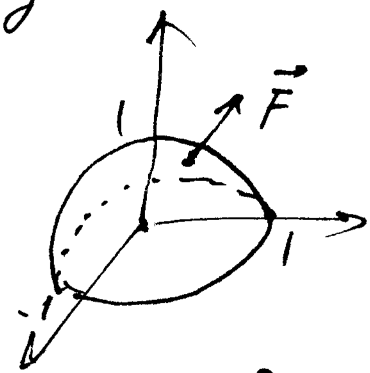


<u>example</u> : Calculate $\iint_S \vec{F} \cdot d\vec{\sigma}$

with $S : x^2+y^2+z^2=1 \ ; \ z \geq 0$

$\vec{F} = x\vec{i} + y\vec{j}$

$$\iint_S \vec{F} \cdot \vec{n} \, d\sigma$$



$$\vec{n} = \frac{\nabla g}{|\nabla g|} \ ; \ g = x^2+y^2+z^2-1$$

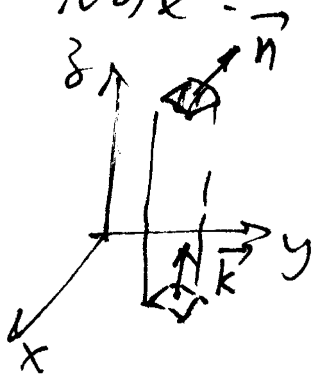
$$= \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{\left((2x)^2+(2y)^2+(2z)^2\right)^{\frac{1}{2}}}$$

$$= x\vec{i} + y\vec{j} + z\vec{k}$$

$$\iint_S \vec{F}\cdot\vec{n}\, d\sigma = \iint (x\vec{i} + y\vec{j})\cdot(x\vec{i} + y\vec{j} + z\vec{k})\, d\sigma$$

$$= \iint_S (x^2 + y^2)\, d\sigma$$

Note:



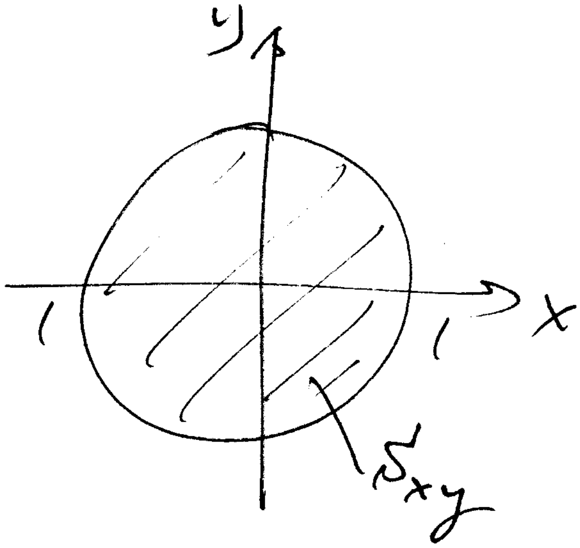
$$d\sigma = \frac{dx\,dy}{z}$$

$$d\sigma(\vec{n}\cdot\vec{k}) = dx\,dy$$

$$d\sigma\cdot z = dx\,dy$$

$$\iint_{S_{xy}} (x^2+y^2)\frac{dx\,dy}{z}$$

S_{xy} ← surface area projected onto the x-y plane



$$z = \sqrt{1-(x^2+y^2)}$$

② $$\iint (x^2+y^2)\frac{dx\,dy}{\sqrt{1-(x^2+y^2)}}$$

③ Using polar coordinates:

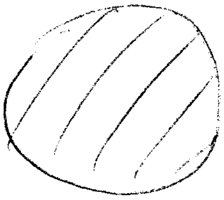
$dx\,dy \longrightarrow r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^1 \frac{r^2 r\,dr\,d\theta}{\sqrt{1-r^2}}$$; Let $u = 1-r^2$

$du = -2r\,dr$

$\sqrt{1-r^2} = u^{\frac{1}{2}}$

$r^2 = 1-u$



$$= \int_0^{2\pi}\int_1^0 \frac{(1-u)\left(-\frac{du}{2}\right)d\theta}{u^{\frac{1}{2}}}$$

$$= \cancel{2}\pi \int \frac{(u-1)}{\cancel{2}u^{\frac{1}{2}}}\,du = \pi\int_1^0 \left[u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right] du$$

$$= \pi \left[\frac{2}{3} u^{\frac{3}{2}} - 2 u^{\frac{1}{2}} \right]_1^0$$

$$= -\pi \left[\frac{2}{3} - 2 \right] = \frac{4}{3}\pi$$

§ <u>Gauss' Theorem</u> (Divergence Theorem)

$$\iint_S \vec{F} \cdot \vec{n} \, d\sigma = \iiint_V \nabla \cdot \vec{F} \, d\tau$$

V is any volume and S is the surface area which encloses V

$\vec{n}$ = unit outward normal vector at S

§ Stokes' Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, d\sigma$$

C : boundary of S