(ii) <u>Differential Operator : D</u>

<u>Def^N</u> :

$$D^1 = D = \frac{d}{dt}$$

$$D^2 = D \cdot D = \frac{d^2}{dt^2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$D^n = \underbrace{D \cdots D}_{n} = \frac{d^n}{dt^n}$$

§ Solution of Non-homogeneous Eq^N

<u>steps</u> :
1. solve the homogeneous Eq^N
2. find any particular Sol^N
3. add the two to get a complete sol^N

1. <u>Method of Undetermined Coeff</u>

The idea here is to generate a <u>family</u> for $f(t)$. A family of fct has the property that repeated differentiation of member of the family produces only member of the family.

Examples of family:

$\sin mt$; $\cos mt$

e^{mt}

$\sinh mt$, $\cosh mt$

$t^p, t^{p-1}, t^{p-2}, \ldots, t^2, t, 1$

Example :

$$\frac{d^2y}{dt^2} + \omega^2 y = t^2$$

homogeneous sol$^{\underline{N}}$: $y_h = \cancel{B}\ A_1 \cos\omega t + A_2 \sin\omega t$

$$\left(\frac{d^2 y_h}{dt^2} + \omega^2 y_h = 0 \right.$$

$\omega = 0$: $y_h = A_1$

$$\frac{d^2 y_h}{dt^2} = 0 \Longrightarrow y_h = A_1 + A_2 t$$

$$y_h = A_1 \cos\omega t + \frac{A_2}{\omega}\sin\omega t$$

$$\lim_{\omega \to 0} \frac{\sin\omega t}{\omega} = \lim_{\omega \to 0} t \, \frac{\sin\omega t}{\omega t}$$

$$= t$$

particular solⁿ : $f(t) = t^2$

$$y_p = C_1 t^2 + C_2 t + C_3$$

$$\frac{dy_p}{dt} = 2C_1 t + C_2$$

$$\frac{d^2 y_p}{dt^2} = 2C_1$$

Sub into D.E.

$$2c_1 + \omega^2(c_1 t^2 + c_2 t + c_3) = t^2$$

$$\Longrightarrow \quad \underline{\omega^2 c_1} t^2 + \omega^2 c_2 t + 2c_1 + c_3 \omega^2 = t^2$$

$$\left.\begin{aligned} t^2 &: \quad \omega^2 c_1 = 1 \\ t &: \quad \omega^2 c_2 = 0 \\ 1 &: \quad c_3 \omega^2 + 2c_1 = 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} c_1 &= \frac{1}{\omega^2} \\ c_2 &= 0 \\ &\cancel{c_3 = 0} \\ c_3 &= \frac{-2c_1}{\omega^2} \\ &= -\frac{2}{\omega^4} \end{aligned}$$

$$\therefore\ y_p = \frac{t^2}{\omega^2} - \frac{2}{\omega^4}$$

$$y = y_h + y_p = A_1 \cos\omega t + A_2 \frac{\sin\omega t}{\omega} + \frac{t^2}{\omega^2} - \frac{2}{\omega^4}$$

Another situation:

$$\frac{d^2y}{dt^2} + \omega^2 y = \sin\omega t$$

let: $y_p = A_1 \sin\omega t + A_2 \cos\omega t$

L.H.S $\quad \frac{d^2y_p}{dt^2} = -A_1\omega^2 \sin\omega t - A_2\omega^2 \cos\omega t$

$$= -\omega^2(A_1 \sin\omega t + A_2 \cos\omega t)$$

$$\omega^2 y_p = \omega^2(A_1 \sin\omega t + A_2 \cos\omega t)$$

$$\frac{d^2 y_p}{dt^2} + \omega^2 y_p = 0$$

$$y_p = t A_1 \sin\omega t + t A_2 \cos\omega t$$

Following the same procedure:

$$y_p = -\frac{t}{2\omega} \cos\omega t$$

$$\left(y_p = \sin\omega t - \frac{t}{2\omega} \cos\omega t \right)$$

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$$y = y_h + y_p = A_1 \sin\omega t + A_2 \cos\omega t - \frac{t}{2\omega}\cos\omega t$$

2. <u>Particular Solution by Reduction of Order</u>

$$(D-m_1)\underbrace{(D-m_2)\cdots(D-m_n)\,y} = f(t)$$

Define: $\eta_1(t) = (D-m_2)\cdots(D-m_n)\,y$

Then: $(D-m_1)\,\eta_1(t) = f(t)$

or $$\frac{d\eta_1}{dt} - m_1\eta_1 = f(t)$$

Integrating factor: $p = e^{\int a_1\,dt}$

$$p = e^{\int -m_1\,dt} = e^{-m_1 t}$$

$$\eta_1(t) = e^{m_1 t}\int e^{-m_1 t} f(t)\,dt + \underbrace{\cancel{c_1}}_{0} e^{m_1 t}$$

for particular solution

$$\eta_2(t) \equiv (D-m_3)\cdots(D-m_n)y$$

$$(D-\underset{m_1}{m_2})\,\underset{\eta_1}{\eta_2}(t) = \underset{f(t)}{\eta_1(t)}$$

Again:

$$\eta_2(t) = e^{m_2 t}\int e^{-m_2 t}\,\eta_1(t)\,dt + \overset{0}{\cancel{c_2}}\, e^{m_2 t}$$

3. Particular Solution by Variation of Parameters

$$L_n y = f(t)$$

n linearly indep. solutions u_j

$$y_h(t) = \sum_{k=1}^{n} A_k u_k(t)$$

For the particular solution, assume

$$y_p = \sum_{k=1}^{n} c_k(t) u_k(t)$$

Now: there are n unknown fcts $c_k(t)$

(ii) <u>Special Classes of 2nd order Eq^N</u>

$$F\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y, x\right) = 0$$

If y or x does not appear explicitly, the following substitutions are useful:

(i) $p = \frac{dy}{dx}$; $\frac{dp}{dx} = \frac{d^2y}{dx^2}$

(ii) $p = \frac{dy}{dx}$; $\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy} \cdot \frac{dy}{dx}$

$$= p\frac{dp}{dy}$$

With them, the eq^N F is