$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0 \quad (m_{1,2} = -1 \pm 2i)$$

$$y = A_1 e^{(-1+2i)t} + A_2 e^{(-1-2i)t}$$

$$A_1 = a_1 + ib_1 \; ; \; A_2 = a_2 + ib_2$$

$$\left(e^{\pm i\theta} = \cos\theta \pm i\sin\theta \right)$$

$$\left[y = e^{-t}\left(c_1\cos 2t + c_2 \sin 2t\right) \right]$$

$$y = (a_1 + ib_1) e^{-t} e^{2it} + (a_2 + ib_2) e^{-t} e^{-2it}$$

$$= \left[(a_1 + ib_1)(\cos 2t + i\sin 2t) + (a_2 + ib_2)(\cos 2t - i\sin 2t) \right] e^{-t}$$

$$= e^{-t}\left[(a_1+a_2)\cos 2t + (-b_1+b_2)\sin 2t\right]$$

$$+\; e^{-t}\, i\left[\underbrace{(b_1+b_2)}_{\shortparallel}\cos 2t + \underbrace{(a_1-a_2)}_{\shortparallel}\sin 2t\right]$$

$b_1 = -b_2$ $\qquad$ $a_1 = a_2$

for y to be real

$\therefore \quad A_1 = a_1 + i b_1$

$\quad\;\; A_2 = a_1 - i b_1$

$$y = e^{-t}\left[C_1 \cos 2t + C_2 \sin 2t\right]$$

C_1 + C_2 are real constants

§ repeated roots

If the characteristic eqn

$$m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0$$

has a twice repeated root m_1, our standard procedure will only generate $n-1$ linearly independent solution of the form

$$y_j(t) = A_j e^{m_j t}$$

Question: How to get the n^{th} solution

Answer: $y = te^{m_1 t}$ is the n^{th} solution

$$\therefore \boxed{\begin{aligned} y_1 &= e^{m_1 t} \\ y_2 &= te^{m_1 t} \end{aligned}}$$

In the case of a root repeated p times, the solution is

$$y = \left(A_1 e^{m_1 t} + A_2 t e^{m_1 t} + A_3 t^2 e^{m_1 t} + \dots + A_p t^{p-1} e^{m_1 t}\right) + A_{p+1} e^{m_{p+1} t} + \dots + A_n e^{m_n t}$$

Example : $\frac{d^3y}{dt^3} - 5\frac{d^2y}{dt^2} + 8\frac{dy}{dt} - 4y = 0$

characteristic
Eqn : $m^3 - 5m^2 + 8m - 4 = 0$

$\Rightarrow (m-1)(m-2)^2 = 0$

$m_{1,2} = 2 \; ; \; m_3 = 1$

$$\boxed{y = A_1 e^{2t} + A_2 t e^{2t} + A_3 e^{t}}$$

§ Equidimensional Equation
(Cauchy's Eqn ; Euler's Eqn)

$$x^n \frac{d^ny}{dx^n} + b_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + b_{n-1} x \frac{dy}{dx} + b_n y = 0$$

Reduce to Eq$^{\underline{N}}$ with ~~coeff~~ constant coeff.
by a change of indep variable.

Let $\quad x = e^z \qquad (z = \ln x)$

$$dx = e^z \, dz$$

$$\frac{dz}{dx} = \frac{1}{e^z} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} = \frac{1}{e^z}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dz}\left(\frac{dy}{dx}\right) \cdot \frac{dz}{dx}$$

$$= \frac{d}{dz}\left(\frac{1}{e^z}\frac{dy}{dz}\right) \cdot \frac{dz}{dx}$$

$$= \left(\frac{1}{e^z}\frac{d^2y}{dz^2} - \frac{1}{e^z}\frac{dy}{dz}\right)\frac{1}{e^z}$$

$$= \frac{1}{e^{2z}}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d}{dz}\left(\frac{d}{dz} - 1\right)y\right)$$

$$\frac{d^m y}{dx^m} = \frac{1}{x^m}\left[\frac{d}{dz}\left(\frac{d}{dz} - 1\right)\left(\frac{d}{dz} - 2\right)\cdots\left(\frac{d}{dz} - m + 1\right)\right]y$$

$$(m = 3: \quad \frac{d^3y}{dx^3} = \frac{1}{x^3}\left[\frac{d}{dz}\left(\frac{d}{dz} - 1\right)\left(\frac{d}{dz} - 2\right)y\right]$$

x-version

y

$x\frac{dy}{dx}$

$x^2\frac{d^2y}{dx^2}$

$\vdots$

$x^m\frac{d^my}{dx^m}$

z-version

y

$\frac{dy}{dz}$

$\frac{d}{dz}\left(\frac{d}{dz}-1\right)y$

$\frac{d}{dz}\left(\frac{d}{dz}-1\right)\cdots\left(\frac{d}{dz}-m+1\right)y$

$\Downarrow$

constant coeff

<u>Example</u> : $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 2y = 0$

Let $x = e^z$; $z = \ln x$

$$\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + 2\frac{dy}{dz} - 2y = 0$$

$$\Rightarrow \frac{d^2y}{dz^2} + \frac{dy}{dz} - 2y = 0$$

Let $y = e^{mz}$ $\Rightarrow$ characteristic eq$\underline{^N}$:

$$m^2 + m - 2 = 0 \Rightarrow m_1 = 1 , m_2 = -2$$

$$y = A_1 e^{m_1 z} + A_2 e^{m_2 z}$$
$$= A_1 e^{z} + A_2 e^{-2z}$$

$$\boxed{y = A_1 x + A_2 x^{-2}}$$

Note: $y = e^{mz} \Longrightarrow y = x^m \quad (x = e^z)$

$$x^2 y'' + 2xy' - 2y = 0$$

$$x^2 (m(m-1)x^{m-2}) + 2x(m x^{m-1}) - 2x^m = 0$$

$$\Rightarrow x^m [\underbrace{m(m-1) + 2m - 2}] = 0$$

$$m^2 + m - 2 \Longrightarrow m = 1, -2$$

$$\therefore y = A_1 x + \frac{A_2}{x^2}$$

In the z-version, a repeated root would give:

$$\left.\begin{aligned} y_1 &= A_1 e^{m_1 z} \\ y_2 &= A_2 z e^{m_1 z} \end{aligned}\right\} \begin{array}{l}\text{with} \\ x = e^{z}\end{array} \quad (z = \ln x$$

$$\Longrightarrow \quad y_1 = A_1 x^{m_1}$$

$$y_2 = A_2 \ln x \, x^{m_1}$$

In the case of a root repeated p times

$$\begin{aligned} y = \; & A_1 x^{m_1} + A_2 \ln x \, x^{m_1} \\ & + A_3 (\ln x)^2 x^{m_1} \\ & + \\ & \vdots \\ & + A_p (\ln x)^{p-1} x^{m_1} + A_{p+1} x^{m_{p+1}} \\ & + \cdots + A_n x^{m_n} \end{aligned}$$

§ Operation Notation :

1. Linear Differential Operator : L_n

$$\underbrace{\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \dots + a_{n-1}\frac{dy}{dt} + a_n y}_{} = f(t)$$

$$\Big[\underbrace{\frac{d^n}{dt^n} + a_1(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_{n-1}\frac{d}{dt} + a_n}_{L_n}\Big] y = f(t)$$