$$x\frac{dy}{dx} - ky = x^2$$

$$\Longrightarrow \frac{dy}{dx} - \frac{k}{x}y = x$$

$$a_1 = -\frac{k}{x} \quad ; \quad f(x) = x$$

$$p(x) = e^{\int a_1 dx} = e^{\int -\frac{k}{x} dx} = e^{-k \ln x}$$

$$= x^{-k}$$

$$\therefore \quad y = x^k \int \frac{x\,dx}{x^k} + Cx^k$$

case 1 : $k = 2$:

$$y = x^2 \ln x + Cx^2$$

Case 2: $k \neq 2$

$$y = x^k \frac{x^{2-k}}{2-k} + c x^k = \frac{x^2}{2-k} + c x^k$$

§ Linear D. E. of Higher Orders

Linear Independence:

Consider the set of fcts: $u_1, u_2, \ldots u_n(t)$

If $\sum_{j=1}^{n} c_j u_j(t) = 0$; c_j are const for $t_1 < t < t_2$

can be satisfied with at least one $c_j \neq 0$, then the set of fcts $u_j(t)$ are linearly dependent.

$$y_j(t) = \frac{1}{c_j}\left[c_1 u_1 + c_2 u_2 + \cdots + c_{j-1} y_{j-1} + c_{j+1} y_{j+1} + \cdots\right]$$

<u>Linear Independence</u> occurs when $\sum_{j=1}^{n} c_j u_j(t) = 0$ is satisfied <u>only</u> by taking all $c_j = 0$.

Assume that $u_j(t)$ are linearly dependent, then:

$$\begin{aligned}
c_1 u_1 + c_2 u_2 + \cdots + c_n u_n &= 0 \\
c_1 u_1' + c_2 u_2' + \cdots + c_n u_n' &= 0 \\
\vdots \qquad \vdots \qquad \vdots \qquad &\vdots \\
c_1 u_1^{n-1} + c_2 u_2^{n-1} + \cdots + c_n u_n^{n-1} &= 0
\end{aligned}$$

In Matrix form

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \vdots & & & \vdots \\ u_1^{n-1} & u_2^{n-1} & \cdots & u_n^{n-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Because at least one $c_j \neq 0$, the determinant of the $n \times n$ matrix must be zero

i.e. $W(u_j) = 0$

Wronskian

$\therefore$ Linear dependent $\Longrightarrow$ $W(u_j) = 0$

$\qquad\qquad\qquad\quad p \qquad\qquad\qquad\qquad\qquad q$

This also gives:

using ~~C~~logic : $p \to q$

$\qquad\qquad\qquad\quad \sim q \to \sim p$

If $W(u_j) \neq 0 \Longrightarrow$ Linear Independent

<u>Note</u>: $W(u_j) = 0 \not\Longrightarrow$ linear dependent

(not necessary)

e.g. $u_j(t) = 1, t, t^2, \ldots, t^n$

$$W(u_j) = \det \begin{vmatrix} 1 & t & t^2 & \cdots & t^n \\ 0 & 1 & 2t & 3t^2 \cdots & n t^{n-1} \\ & & 2 & 6t \cdots & n(n-1) t^{n-2} \end{vmatrix}$$

$$W(y_j) = \det \begin{vmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 2! & & & \\ & & & 3! & & \\ & 0 & & & \ddots & \\ & & & & & n! \end{vmatrix}$$

$$= 2!\,3! \cdots n! \neq 0$$

$\therefore\ 1, t, t^2, \ldots, t^n$ are linearly independent.

§ Existence & Representation Theorem

$$a_0(t)\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}(t)\frac{dy}{dt}$$

$+ y\, a_n(t) = 0$

in which $a_0(t), a_1(t), \ldots, a_n(t)$ are ~~continuous~~
continuous fcts of t
in some interval I of t.
and also $a_0(t) \neq 0$

(a) The eqn has n linearly independant solutions for t in I

If $y_1(t), y_2(t), \ldots, y_n(t)$ are ~~the~~ n linearly independent solutions

then $y = \sum_{j=1}^{n} c_j\, y_j(t)$ is also a

soln. Conversely every solution of

the equation can be represented by an appropriate choice of the constants.

(b) The inhomogeneous form

$$\sum_{k=0}^{n} a_k(t) \frac{d^{n-k} y}{dt^{n-k}} = f(t) \qquad (1)$$

where $f(t)$ is continuous in I

If y_p is a solution to the nonhomogeneous eqn, then

$$y(t) = y_p + \sum_{j=1}^{n} c_j y_j(t)$$

is a solution to (1)

(c) For every point t_0 in I, and for every set of constants k_j, $j=1,2,\dots,n$ there is one and only one solution $y(t)$ to ① satisfying the condition

$$y(t_0)=k_1$$
$$y'(t_0)=k_2$$
$$\vdots$$
$$y^{n-1}(t_0)=k_n$$

§ Case of Constant Coefficient

$$y^n + a_1 y^{n-1} + \dots a_{n-1} y' + a_n y = 0$$

Try $y = Ae^{mt}$

$$Ae^{mt}\left[m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n\right] = 0$$

Since $A \neq 0$ & $e^{mt} \neq 0$, we have left the "characteristic eq^n"

$$\boxed{m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0}$$

In general, we have n roots;

$$m_j \, ; \quad j = 1, 2, \ldots, n$$

$$y_j = A_j e^{m_j t}$$

& $$y(t) = A_1 e^{m_1 t} + A_2 e^{m_2 t} + \cdots + A_n e^{m_n t}$$

$$\cancel{(m-1)^2 = 0 \quad m=1}$$

(1) Complex roots :

$$m_1 = a + ib$$
$$m_2 = a - ib$$

Euler formula : $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

e.g. $$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0$$

Let $y = e^{mt}$

$$m^2 + 2m + 5 = 0$$

$$m_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot 5}}{2} = -1 \pm 2i$$

$$y = A_1 e^{(-1+2i)t} + A_2 e^{(-1-2i)t}$$

Let $A_1 = a_1 + ib_1$

$A_2 = a_2 + ib_2$