Ordinary Differential Equations

§ Basic Nomenclature

$$a_0(y,t)\frac{d^n y}{dt^n} + a_1(y,t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_{n-1}(y,t)\frac{dy}{dt}$$
$$+ a_n(y,t)y = f(t)$$

t : indep variable
y : dependent variable

- If $f(t)=0$; the equation is homogeneous
  $f(t) \neq 0$; " " " nonhomogeneous

n is the order of the D.E.; the order of the highest derivative

The ~~a~~ equation is linear if a's all depend only on t, or are constants

The equation is ordinary if it contains only ordinary derivatives

e.g. $\frac{d^2y}{dt^2} + \frac{dy}{dt} + 3y = 2t$ (O.D.E.)

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \quad \text{(P.D.E)}$$

A solution is any functional relation that satisfies the D.E. in a form not containing integrals or derivatives of unknown fcts.

§ <u>Solution of 1^{st} order equations</u>

For $n=1$, the homogeneous equation is.:

$$a_0(y,t)\frac{dy}{dt} + a_1(y,t)\,y = 0$$

Rearranging & renaming things:

$$F(x,y)\,dx + G(x,y)\,dy = 0$$

1. <u>Exact Differential</u>

Consider $U(x,y) = C$

$$\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy = 0$$

Comparing the two equations:

$$F(x,y) = \frac{\partial U}{\partial x} \quad ; \quad G(x,y) = \frac{\partial U}{\partial y}$$

Note:

$$\frac{\partial F}{\partial y} = \frac{\partial^2 U}{\partial y \partial x} = \frac{\partial G}{\partial x} = \frac{\partial^2 U}{\partial x \partial y}$$

If it is an exact differential then $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$ & we can integrate $\frac{\partial U}{\partial x}$ & $\frac{\partial U}{\partial y}$ to get U.

e.g. Find a sol^{n} to

$$\frac{2xy+1}{y}\,dx + \frac{y-x}{y^2}\,dy = 0$$

$$\underset{F(x,y)}{\overset{\shortparallel}{\frac{2xy+1}{y}}} \qquad G(x,y)$$

$$\frac{\partial F}{\partial y} = -\frac{1}{y^2} \overset{?}{=} \frac{\partial G}{\partial x} = -\frac{1}{y^2}$$

Yes

$\therefore$ it is an exact differential

$$F = \frac{\partial U}{\partial x} \qquad \Big| \qquad G = \frac{\partial U}{\partial y}$$

$$\Longrightarrow U = \int F\,dx + f_1(y) \qquad \Big| \qquad U = \int G\,dy + f_2(x)$$

$$U = \int \frac{2xy+1}{y}\,dx + f_1(y)$$
$$= \frac{x^2y + x}{y} + f_1(y)$$
$$= \underbrace{x^2 + \frac{x}{y}}_{} + f_1(y)$$

$$U = \int \frac{y-x}{y^2}\,dy + f_2(x)$$
$$= \int \left(\frac{1}{y} - \frac{x}{y^2}\right) dy + f_2(x)$$
$$= \underbrace{\ln y + \frac{x}{y}}_{} + f_2(x)$$

Comparing both sides : $f_1(y) = \ln y$

$f_2(x) = x^2$

$$U = x^2 + \frac{x}{y} + \ln y \qquad U = \ln y + \frac{x}{y} + x^2$$

$$\Rightarrow \boxed{x^2 + \frac{x}{y} + \ln y = C}$$

§ Use of an integrating factor

If $F(x,y)\,dx + G(x,y)\,dy$ is <u>not</u> an exact differential (i.e. $\frac{\partial F}{\partial y} \neq \frac{\partial G}{\partial x}$) it can be made one by multiplying by an integrating factor, $p(x)$.

$$p(x)F(x,y)\,dx + p(x)\,G(x,y)\,dy$$

may then be an exact differential (

e.g. Consider $x\,dy - y\,dx$

$\cancel{F}$ $\cancel{G}$

$\cancel{\frac{\partial F}{\partial}}$

$$\left.\begin{array}{l} F = -y \\ G = x \end{array}\right\} \frac{\partial F}{\partial y} = -1 \neq \frac{\partial G}{\partial x} = 1$$

$\therefore$ not an exact differential.

observe:

$$d\left(\frac{y}{x}\right) = \underbrace{\frac{1}{x}}_{G} dy \underbrace{-\frac{y}{x^2}}_{F} dx$$

$$\frac{\partial G}{\partial x} = -\frac{1}{x^2} = \frac{\partial F}{\partial y} = -\frac{1}{x^2}$$

$$= \frac{x\,dy - y\,dx}{x^2}$$

$\therefore$ If we multiply by the integrating factor $\frac{1}{x^2}$, the result is an exact differential

we have $\cancel{\frac{d}{dx}}\ d\left(\frac{y}{x}\right) = 0$

$$\Longrightarrow \quad \frac{y}{x} = c \Longrightarrow \boxed{y = cx}$$

§ Separation of Variable

$$\underbrace{F(x,y)dx + G(x,y)\,dy}_{\cancel{x\,dy - y\,dx}} = 0$$

If F and G are separable

i.e. $F(x,y) = F_1(x)\,F_2(y)$

$\quad\;\; G(x,y) = G_1(x)\,G_2(y)$

then the solution is simple.

$$F_1(x)F_2(y)\,dx + G_1(x)G_2(y)\,dy = 0$$

$$\Longrightarrow \quad \frac{F_1(x)}{G_1(x)}\,dx + \frac{G_2(y)}{F_2(y)}\,dy = 0$$

$$\Longrightarrow \quad \int \frac{F_1(x)\,dx}{G_1(x)} + \int \frac{G_2(y)}{F_2(y)}\,dy = C$$

§ <u>Change of variables</u>

Def$^{\underline{n}}$: A fct $F(x,y)$ is homogeneous to degree n if there is a constant n, s.t. for every λ

$$F(\lambda x, \lambda y) = \lambda^n F(x,y)$$

e.g. $F(x,y) = x^6 + 2x^2y^4 + xy^5$

If $F(x,y)$ & $G(x,y)$ are both homogeneous to degree n, then the substitution: $y = vx$

or $x = uy$

leads to a separable equation.

Let $y = vx \implies dy = v\,dx + x\,dv$

D.E. becomes:

$$F(x, vx)\,dx + G(x, vx)\underbrace{[v\,dx + x\,dv]}_{dy} = 0$$

$$\cancel{x^n} F(1, v)\,dx + \cancel{x^n} G(1, v)[v\,dx + x\,dv] = 0$$

$$\implies [F(1, v) + v\,G(1, v)]\,dx = -x\,G(1, v)\,dv$$

$$\Rightarrow \int \frac{dx}{x} = -\int \frac{G(1,v)\,dv}{F(1,v)+v\,G(1,v)} + C$$

$$\Rightarrow \ln x = \qquad v \qquad + C$$

e.g. Solve $(y^2 - xy)dx + x^2 dy = 0$

$$\underbrace{y^2-xy}_{F} \qquad \underbrace{x^2}_{G}$$

$$\frac{\partial F}{\partial y} = 2y - x \neq \frac{\partial G}{\partial x} = 2x$$

$\therefore$ not an exact differential

By inspection, F & G are homogeneous to $n = 2$; let $y = vx$

$$(v^2x^2 - x \cdot vx)dx + x^2[v\,dx + x\,dv] = 0$$

$$x^2(v^2 - v)dx + x^2[v\,dx + x\,dv] = 0$$

$$x^2 v^2\,dx = -x^3\,dv$$

$$\Longrightarrow \int \frac{dx}{x} = -\int \frac{dv}{v^2} + C$$

$$\Longrightarrow \ln x = \frac{1}{v} + C$$

$$\Longrightarrow \boxed{\ln x = \frac{x}{y} + C} \qquad (y = vx \quad v = \frac{y}{x}$$

§ Linear Equations of First order

$$a_0^*(x)\frac{dy}{dx} + a_1^*(x)\,y = g(x)$$

$\div\ a_0^*(x)$:

$$\boxed{\frac{dy}{dx} + a_1(x)\,y = f(x)}$$

Multiply by an integrating factor $p(x)$

$$p(x)\frac{dy}{dx} + p(x)\,q_1(x)\,y = p(x) f(x) \qquad (1)$$

Compare with the exact differential

$$\frac{d}{dx}[p(x)y] = p(x)\frac{dy}{dx} + y\frac{dp}{dx}$$

L.H.S. of (1) will be an exact differential if: $\frac{dp}{dx} = p(x)\,q_1(x)$

$$\Longrightarrow \boxed{p(x) = e^{\int q_1(x)\,dx}}$$

Eq (1) becomes:

$$p(x)\frac{dy}{dx} + p(x)\,a_1(x)\,y = \frac{d}{dx}[p(x)\,y]$$
$$= p(x)\,f(x)$$

$$\Longrightarrow \quad p(x)\,y = \int p(x)\,f(x)\,dx + C$$

$$\Longrightarrow \quad \boxed{y = \frac{1}{p(x)}\int p(x)\,f(x)\,dx + \frac{C}{p(x)}}$$

nonhomogeneous solution (particular solution)   homogeneous sol$^{\underline{n}}$

$$y = e^{-\int a_1 dx}\int e^{\int a_1 dx} f(x)\,dx + C e^{-\int a_1 dx}$$

eg. 5 $\quad x\,dy - y\,dx = 0$

$$\left[\ y = cx\ \right]$$

$$\frac{dy}{dx} - \frac{y}{x} = 0$$

$$a_1 = -\frac{1}{x}, \quad f(x) = 0$$

$$p(x) = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

$$\cancel{\frac{1}{x}}\,\frac{dy}{dx} - \frac{y}{x} = 0$$

$$\Longrightarrow \quad \cancel{\frac{d}{dx}(xy) = 0} \quad \frac{d}{dx}\left(\frac{y}{x}\right) = 0$$

$$\Longrightarrow \quad \frac{y}{x} = c$$